Basic Matrix Operations

Definition: A m row by n column matrix A is a rectangular array of mn entries

$$A = \begin{array}{cc} & \text{column 2} \\ \text{row 1} \rightarrow & \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{array} \qquad a_{ij} \tag{1}$$

We call a m row by n column matrix a $m \times n$ matrix.

Definition: Let A, B be $m \times n$ matrices and c a real number in $\mathbb{R}$ where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \qquad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}$$

We define

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{bmatrix}, \qquad cA = \begin{bmatrix} ca_{11} & ca_{12} & \cdots & ca_{1n} \\ ca_{21} & ca_{22} & \cdots & ca_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ca_{m1} & ca_{m2} & \cdots & ca_{mn} \end{bmatrix}$$

and

$$-A = (-1)A, \qquad A - B = A + (-B)$$

Example 1: Suppose $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 & 3 \\ -4 & -1 & -5 \end{bmatrix}$. Calculate $-A$ and $2A + B$.

$$-A = (-1)A = \begin{bmatrix} -1 & 0 & 1 \\ -2 & -2 & -3 \end{bmatrix}$$

$$2A + B = \begin{bmatrix} 2 & 0 & -2 \\ 4 & 4 & 6 \end{bmatrix} + \begin{bmatrix} 2 & -1 & 3 \\ -4 & -1 & -5 \end{bmatrix} = \begin{bmatrix} 4 & -1 & 1 \\ 0 & 3 & 1 \end{bmatrix}$$

Definition: We call

(a square matrix)

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2}$$

the $n \times n$ identity matrix.

Example 2: Let $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$. Calculate $A - 2I_3$.

$$A - 2I_3 = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Definition: Let $A = [a_{ij}]$ be a $m \times n$ matrix. We define the transpose of A, denoted A^T, to be the $n \times m$ matrix defined using the formula

$$(A^T)_{ij} = a_{ji} \qquad (A^T)_{12} = a_{21} \tag{3}$$

Example 3: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$. Calculate A^T and $(A^T)^T$. What do you observe?

2×3 $\qquad$ 3×2

$$A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \qquad (A^T)^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

Theorem 1 (Poole 3.4 part a): If A is a $m \times n$ matrix then

$$(A^T)^T = A \tag{4}$$

Example 4: If $\mathbf{x} = \underset{n \times 1}{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}$, then $\mathbf{x}^T = \underset{1 \times n}{\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}}$ is called a *row vector*.

Example 5: Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. Calculate $(A+B)^T$ and $A^T + B^T$. Observation?

$A + B = \begin{bmatrix} 2 & 2 \\ 3 & 2 \end{bmatrix}$ $\quad (A+B)^T = \begin{bmatrix} 2 & 3 \\ 2 & 2 \end{bmatrix}$

$A^T = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$ $\quad B^T = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

$A^T + B^T = \begin{bmatrix} 2 & 3 \\ 2 & 2 \end{bmatrix}$ $\quad (A+B)^T = A^T + B^T$

Theorem 2 (Poole 3.4 parts b and c): If A and B are $m \times n$ matrices and k in $\mathbb{R}$ then

$$(A+B)^T = A^T + B^T, \qquad (kA)^T = k(A^T) \tag{5}$$

Definition: A $n \times n$ matrix A is called symmetric if $A^T = A$.

Example 6: Which of the following matrices are symmetric? Explain.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 3 & -1 & 2 \\ -1 & 1 & 4 \\ 2 & 4 & 5 \end{bmatrix} \tag{6}$$

$A^T = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix}$ Since $A^T \neq A$, A is not symmetric

$B^T = \begin{bmatrix} 3 & -1 & 2 \\ -1 & 1 & 4 \\ 2 & 4 & 5 \end{bmatrix}$ Since $B^T = B$, B is symmetric